Interlaminar Stresses in Laminated Cylindrical Shells of Composite Materials

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Many existing analysis methods for laminated composite material structural components assume structural integrity between laminae and predict load carrying capabilities considerably higher than those sometimes obtained experimentally. These methods do not include the means to determine accurately the interlaminar shear and normal stresses, which can cause premature structural failure. Methods of analysis are presented herein for a laminated circular cylindrical shell composed of generally orthotropic materials subjected to arbitrary axially symmetric loadings. Elastic shell theory, including transverse shear deformation, is utilized. By treating each lamina individually, in conjunction with imposing stress and displacement boundary conditions between laminae, the governing equations for the individual laminae are combined, yielding the interlaminar shear and normal stresses as explicit dependent variables. A parametric study for a generally othotropic circular cylindrical shell, subjected to a uniform pressure with both clamped and simple supported end boundary conditions is presented. Variables include laminate geometry, fiber orientation, stacking sequence and material properties. The solutions obtained provide insight into the interlaminar stress fields of shells with any number of plies, and provide a baseline for finite element and finite difference solutions for the same problem. A computer program has been formulated to expedite calculations with the analytical solution.

I. Introduction

AT the present time many of the existing methods of analysis for laminated composite structures are direct extensions of classical plate and shell structures. However, Pagano[1] and Wu and Vinson[2] have shown that transverse shear deformation must be accounted for in the analysis of any thin beam, plate, or shell structure. Inclusive reviews of anistropic plate and shell research have been made by Kingsbury,[3] Wu,[4] Daugherty[5] and Grigolyuk.[6] Even including transverse shear deformation, existing analysis methods which consider laminated constructions as one equivalent single layer, such as describing the stiffnesses by the *A, B,* and *D* matrices, sometimes predict load carrying capabilities considerably higher than those determined experimentally. Compared with analyzing single layered plates and shells of either isotropic or orthotropic materials, the analysis of multi-layered plates and shells, which account for the determination of shear and normal stresses between laminae, is a relatively unexplored area.

Concerning analyses accounting for interlaminar stresses, Summers and Vinson[7] in 1966 analyzed the interlaminar stresses occurring in rectangular orthotropic laminated plates. In 1968 Franklin and Kicher[8] analyzed stresses in a laminated composite circular cylinder of three layers, wherein the middle layer is isotropic. However, their analysis included neither transverse shear deformation nor bending effects.

In 1969 Petit[9] and Ashton[10] studied interlaminar shear stresses in the vicinity of the free edges in laminates. In 1970, Pipes and Pagano[11] used elasticity theory to study the behavior of a finite width composite laminate under uniform axial strain. Using a finite difference technique, they provided evidence that the interlaminar shear stresses mathematically have a singularity at the intersection of the interface between laminae and the free edge of the laminate.

Next, Pagano and Pipes[12] studied the effect of stacking sequence on laminate strength, in order to prevent delamination under uniaxial static and fatigue loads. In 1971, Pipes and Daniel[13] verified that in laminates the interlaminar free edge effect is confined to a boundary-layer region approximately equal to the laminate thickness.

Employing analysis methods developed by Summers and Vinson[7] Mehta[14] and Zukas[15] obtained solutions for interlaminar stresses occurring in multilayer plate and shell structures composed of pyrolytic graphite-type materials subjected to thermal loads.

In 1972, Whitney and Browning[16] presented experimental results which clearly showed the free edge delamination of orthotropic symmetrically laminated composites subjected to both in-plane uniaxial static and fatigue loads. Then Pipes[17] studied in depth the interlaminar stresses in a composite plate subjected to in-plane loads, including the effects of stacking sequence, laminate geometry, material properties, and fiber orientation.

In 1974, Pagano[18] developed an approximate method to determine the distribution of the interlaminar normal stresses in the central plane of a symmetric, finite width composite flat laminate. These methods modified a higher order theory of Whitney and Sun[19] for extensional motion and they agree with the three-dimensional elasticity solution of Pipes. Finally, in 1974, Lackman and Pagano[20] developed a practical approach to modify localized geometry of a flat composite laminate to minimize failure by delamination, which is applicable to static or fatigue loadings. It is hoped that the following methods will add insight and rigor to the analysis of composite material laminated shells, since their behavior has some distinctive differences compared to flat laminates.

II. Derivation of Governing Equations

The equilibrium equations and the linear strain-displacement relations for a circular cylindrical shell subjected to axially symmetric loadings conditions, can be found in many references.[5,21,22] The constitutive relations for a shell lamina whose principal material axes (denoted by 1 and 2) are inclined from the shell principal axes (denoted by x and θ) by

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an angle ϕ (measured such that the 1^+ axis is rotated from the x^+ axis in the direction of the θ^+ axis) can be written as

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{\theta z}\\ \sigma_{xz}\\ \sigma_{x\theta}\end{bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & 0 & 2\bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & 0 & 2\bar{Q}_{26}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\bar{Q}_{44} & 2\bar{Q}_{45} & 0\\ 0 & 0 & 0 & 2\bar{Q}_{45} & 2\bar{Q}_{55} & 0\\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & 0 & 2\bar{Q}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \epsilon_{\theta z}\\ \epsilon_{xz}\\ \epsilon_{x\theta}\end{bmatrix} \tag{1}$$

where

$$\bar{Q}_{11} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4$$

$$\bar{Q}_{22} = Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(n^4 + m^4)$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})nm^3 + (Q_{12} - Q_{22} + 2Q_{66})n^3m$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})n^3m + (Q_{12} - Q_{22} + 2Q_{66})nm^3$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(n^4 + m^4)$$

$$\bar{Q}_{44} = Q_{44}m^2 + Q_{55}n^2, \quad \bar{Q}_{55} = Q_{44}n^2 + Q_{55}m^2$$

$$\bar{Q}_{45} = (Q_{55} - Q_{44})mn, \quad m = \cos\phi, \quad n = \sin\phi$$

and the elastic strains ϵ_{ij} are related to displacements u_i in a three-dimensional body by

$$\epsilon_{ij} = ½(u_{i,j} + u_{j,i}) \quad i,j = x,\theta,z \tag{2}$$

The inclusion of the coefficient "2" in some of the terms in (1) merely is to define the Q_{ij} terms exactly, as done by Ashton et al.[23] and most researchers since then.

The customary shell theory assumptions are made,[21,22] including Love's First Approximations, $(h/R_{\min}) \ll 1$, where h is the shell thickness and $R_{\min}$ is the cylindrical shell radius; but transverse shear deformation is retained because in most filamentary composites, $20 < E_{11}/G_{13} < 50$, where subscript 1 refers to the fiber direction. The form of the displacements in the lamina are

$$u_x = u_0(x) + z\beta_x(x), \quad u_\theta = v_0(x) + z\beta_\theta(x), \quad u_z = w(x) \tag{3}$$

where u_x, u_θ, u_z are the displacements in the x, θ, and z direction, respectively, u_0 and v_0 are the midsurface displacements in the x and θ directions, z is the distance from the lamina midplane, and β_x and β_θ are the rotations. Defining the usual stress resultants,[21] stress couples, and shear resultants, the equilibrium equations are found to be

$$(dN_x/dx) + q_x = 0, \quad (dN_{x\theta}/dx) + (Q_\theta/R) + q_\theta = 0 \tag{4a}$$

$$(dQ_x/dx) - (N_\theta/R) + p = 0, \quad (dM_x/dx) - (Q_x - m_x) = 0 \tag{4b}$$

$$(dM_{x\theta}/dx) - (Q_\theta - m_\theta) = 0 \tag{4c}$$

where the q_i, p, and m_i are the net surface shear loading, normal loading, and surface shear stress couples per unit mid plane surface defined by

$$q_i \equiv \sigma_{iz}(+h/2) - \sigma_{iz}(-h/2), \quad (i = x,\theta)$$

$$m_i \equiv h/2[\sigma_{iz}(+h/2) + \sigma_{iz}(-h/2)], \quad (i = x,\theta)$$

$$p \equiv \sigma_{zz}(+h/2) - \sigma_{zz}(-h/2)$$

From the foregoing, the integrated resultant and stress couple relations in terms of displacements are found to be[24]

$$N_x = \bar{Q}_{11}hDu_0 + \bar{Q}_{12}(h/R)w + \bar{Q}_{16}hDv_0 \tag{5a}$$

$$N_\theta = \bar{Q}_{12}hDu_0 + \bar{Q}_{22}(h/R)w + \bar{Q}_{26}hDv_0 \tag{5b}$$

$$N_{x\theta} = \bar{Q}_{16}hDu_0 + \bar{Q}_{26}(h/R)w + \bar{Q}_{66}hDv_0 \tag{5c}$$

$$M_x = (1/12)\hat{Q}_{11}h^3D\beta_x + (1/12)\bar{Q}_{16}h^3D\beta_\theta \tag{5d}$$

$$M_\theta = (1/12)\bar{Q}_{12}h^3D\beta_x + (1/12)\bar{Q}_{26}h^3D\beta_\theta \tag{5e}$$

$$M_{x\theta} = (1/12)\bar{Q}_{16}h^3D\beta_x + (1/12)\bar{Q}_{66}h^3D\beta_\theta \tag{5f}$$

where

$$D(\ ) = d(\ )/dx$$

Following Daugherty,[5] the integrated transverse shear resultant expressions are found to be

$$Q_x = (m_x/6) + (5/6)h\{\bar{Q}_{45}[\beta_\theta - (v_0/R)] + \bar{Q}_{55}(\beta_x + Dw)\} \tag{6a}$$

$$Q_\theta = (m_\theta/6) + (5/6)h\{\bar{Q}_{44}[\beta_\theta - (v_0/R)] + \bar{Q}_{45}(\beta_x + Dw)\} \tag{6b}$$

Finally, the five governing equations for the k^{th} lamina of a generally orthotropic circular cylindrical thin shell subjected to axially symmetric loadings, including transverse shear deformation are

$$\bar{Q}_{11}hD^2u_0 + \bar{Q}_{12}\frac{h}{R}Dw + \bar{Q}_{16}hD^2v_0 + q_x = 0 \tag{7}$$

$$\frac{1}{12}\bar{Q}_{11}h^3D^3\beta_x + \frac{1}{12}\bar{Q}_{16}h^3D^3\beta_\theta - \bar{Q}_{12}\frac{h}{R}Du_0$$

$$-Q_{22}\frac{h}{R^2}w - \bar{Q}_{26}\frac{h}{R}Dv_0 + Dm_x + p = 0 \tag{8}$$

$$\frac{1}{12}\bar{Q}_{11}h^3D^2\beta_x + \frac{1}{12}\bar{Q}_{16}h^3D^2\beta_\theta - \frac{5}{6}h$$

$$\times\left[\bar{Q}_{45}\left(\beta_\theta - \frac{v_0}{R}\right) + \bar{Q}_{55}(\beta_x + Dw)\right] + \frac{5}{6}m_x = 0 \tag{9}$$

$$\frac{1}{12}\bar{Q}_{16}\frac{h^3}{R}D^2\beta_x + \frac{1}{12}\bar{Q}_{66}\frac{h^3}{R}D^2\beta_\theta + \bar{Q}_{16}hD^2u_0$$

$$+ Q_{26}\frac{h}{R}Dw + \bar{Q}_{66}hD^2v_0 + q_\theta = 0 \tag{10}$$

$$\frac{1}{12}\bar{Q}_{16}h^3D^2\beta_x + \frac{1}{12}\bar{Q}_{66}h^3D^2\beta_\theta - \frac{5}{6}h$$

$$\times\left[\bar{Q}_{44}\left(\beta_\theta - \frac{v_0}{R}\right) + \bar{Q}_{45}(\beta_x + Dw)\right] + \frac{5}{6}m_\theta = 0 \tag{11}$$

Note that when treating multilayer shells, each symbol above should have subscript "k". Hence, in treating a laminated shell of n laminae, one must solve $5n$ coupled ordinary linear differential equations. To illustrate the technique of using the above governing equations consider the two-lamina case.

III. Two-Lamina Composite Shell

Equations (7) through (11) are written for both A and B laminae, as shown in Fig. 1. Note that the subscripts 1 and 2 on all stresses on the lateral surfaces of the shell refer to those

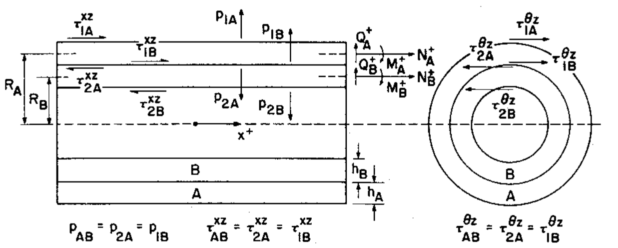

Fig. 1 Forces on a two-lamina composite shell.

on the outer and inner surfaces, respectively. Thus, in this case, p_{iA}, p_{2A},

$$\tau_{1A}^{xz}, \quad \tau_{1A}^{\theta z}, \quad \tau_{2B}^{xz} \quad \text{and} \quad \tau_{2B}^{\theta z}$$

are the loads on the lateral surfaces prescribed in any deterministic shell problem.

At the interface it is clear that, from equilibrium considerations, the following must be true:

$$\sigma_{xz}(-h_A/2) = \sigma_{xz}(+h_B/2) \equiv \tau_{AB}^{xz} \tag{12a}$$

$$\sigma_{\theta z}(-h_A/2) = \sigma_{\theta z}(+h_B/2) \equiv \tau_{AB}^{\theta z} \tag{12b}$$

$$\sigma_{zz}(-h_A/2) = \sigma_{zz}(+h_B/2) \equiv p_{AB} \tag{12c}$$

Also in Fig. 1, the conditions that the laminae remain bonded together and that no slippage occurs in the joint between laminae require that:

$$w_A = w_B \tag{13a}$$

$$u_x(-h_A/2) = u_x(+h_B/2) \tag{13b}$$

$$u_\theta(-h_A/2) = u_\theta(+h_B/2) \tag{13c}$$

From (3), the latter two require that

$$u_{oA} - (h_A/2)\beta_{xA} = u_{0B} + (h_B/2)\beta_{xB} \tag{14a}$$

$$v_{0A} - (h_A/2)\beta_{\theta a} = v_{0B} + (h_B/2)\beta_{\theta B} \tag{14b}$$

Thus, Eqs. (7) through (11) written, once with subscripts A, once with subscripts B, and employing the joint integrity relations (12) through (14) above, result in ten equations.

From these ten equations, explicit expressions for the joint normal and shear stresses result,[24] as shown below:

$$P_{AB} = P_{2B} - \frac{h_B}{2} D\tau_{2B}^{xz} - \frac{1}{12}\bar{Q}_{11_B} h_B^3 D^3 \beta_{xB} - \frac{1}{12}\bar{Q}_{16_B} h_B^3 D^3 \beta_{\theta B} - \frac{h_B}{2} D\tau_{AB}^{xz} + \bar{Q}_{12_B}\frac{h_b}{R_b} Du_{0B} + \bar{Q}_{26_B}\frac{h_B}{R_B} Dv_{0B} + \bar{Q}_{22_B}\frac{h_B}{R_B^2} w_B \tag{15}$$

$$\tau_{AB}^{xz} = \tau_{2B}^{xz} - \bar{Q}_{11_B} h_B D^2 u_{0B} - \bar{Q}_{16_B} h_B D^2 v_{0B} - \bar{Q}_{12_B}\frac{h_B}{R_B} Dw_B \tag{16}$$

$$\tau_{AB}^{\theta z} = \tau_{1A}^{\theta z} + \bar{Q}_{16_A} h_A D^2\left(u_{0B} + \frac{h_A}{2}\beta_{xA} + \frac{h_B}{2}\beta_{xB}\right) + \bar{Q}_{66_A} h_A D^2 \times \left(v_{0B} + \frac{h_A}{2}\beta_{\theta A} + \frac{h_B}{2}\beta_{\theta B}\right) + \bar{Q}_{26_A}\frac{h_A}{R_A} Dw_B \tag{17}$$

These can be solved for explicitly upon the solution of the total problem. Analogous expressions can be obtained for shells of a greater number of laminae.

IV. Solutions

The solution to the $5n$ set of equations are complex exponential functions with complex constant coefficients. In the two-lamina case, the general form of the equations can be summarized as follows:

$$D^4(\alpha_1 D^{10} + \alpha_2 D^8 + \alpha_3 D^6 + \alpha_4 D^4 + \alpha_5 D^2 + \alpha_6)\begin{bmatrix}\beta_{xA}\\ \beta_{xB}\\ \beta_{\theta A}\\ \beta_{\theta B}\\ w_B\\ v_{oB}\\ u_{oB}\end{bmatrix} = \begin{bmatrix}\alpha_7\\ \alpha_8\\ \alpha_9\\ \alpha_{10}\\ \alpha_{11}\\ \alpha_{12}\\ \alpha_{13}\end{bmatrix} \tag{18}$$

wherein the α_i are constants. The other three variables, Eqs. (15-17), are determined once the previous set of seven is solved. The details of the solution are given in Ref. 24. The general solutions contain a total of fourteen real constants which are determined by seven boundary conditions for the laminated shell at each end.

V. Boundary Conditions

By the principal of virtual displacements, the required boundary conditions at the end of the shell[24] are found to be the following for the two-lamina-shell:

Either $\bar{N}_x = N_{xA} + N_{xB} = 0$ or u_{0B} is prescribed.

Either $\bar{N}_{x\theta} = N_{x\theta A} + N_{x\theta B} = 0$ or v_{0B} is prescribed.

Either $\bar{Q}_x = Q_{xA} + Q_{xB} = 0$ or $w_B(=w_A)$ is prescribed.

Either $\bar{M}_{xA} = M_{xA} + ½h_A N_{xA} = 0$ or β_{xA} is prescribed.

Either $\bar{M}_{xB} = M_{xB} + ½h_B N_{xA} = 0$ or β_{xB} is prescribed.

Either $\bar{M}_{x\theta A} = M_{x\theta A} + ½h_A N_{x\theta A} = 0$ or $\beta_{\theta A}$ is prescribed.

Either $\bar{M}_{x\theta B} = M_{x\theta B} + ½h_B N_{x\theta A} = 0$ or $\beta_{\theta B}$ is prescribed.

It is straightforward to obtain analogous expressions for shells of "n" laminae, and the boundary conditions for the three-lamina case are given in Ref. 24. From the boundary conditions just given, it is clear which choices to utilize for simply supported, clamped, and free boundary conditions.

VI. Parametric Study

To investigate the effects of several variables on the interlaminar shear and normal stresses, a parametric study is presented considering both two- and three-laminae circular cylindrical shells subjected to a constant internal pressure for both clamped and simply supported conditions. The variables involved include laminate geometry, fiber orientation, stacking sequences, and material properties.

The baseline material is boron-epoxy, with the following material properties: $E_{11} = 32.5\text{x}10^6$ psi, $E_{22} = 1.84 \times 10^6$ psi, $\nu_{12} = 0.256$, $G_{12} = G_{13} = 0.642 \times 10^6$ psi, $G_{23} = 0.361 \times 10^6$ psi. The baseline cylindrical shell geometry is $h = 0.25$ in., $R = 12.5$ in., $L = 50$ in. The internal pressure is $p = 100$ psi.

Figures 2-4 show the interlaminar axial shear stress, circumferential shear stress, and normal stress distribution for a $+45°/-45°$ angle ply composite shell with both clamped and simply supported end conditions, for the region of shell midlength ($x = 0$) to one end ($x = 25$ in). For both the clamped and simply supported cases, the interlaminar axial shear stress τ_{AB}^{xz} is the maximum at the end of the shell. On the other hand, the circumferential shear stress $\tau_{AB}^{\theta z}$ is a maximum at the end for the simply supported case and peaks near the end for the clamped case. In each case the interlaminar shear stresses decrease rapidly to zero a short distance away from the shell edge.

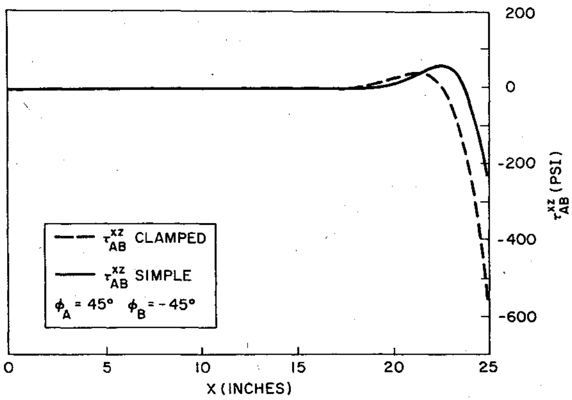

Fig. 2 Interlaminar axial stress distribution for both clamped and simply supported ends.

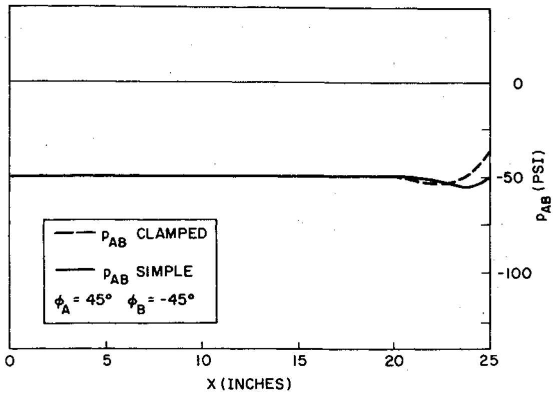

Fig. 4 Interlaminar normal stress distribution for both clamped and simply supported ends.

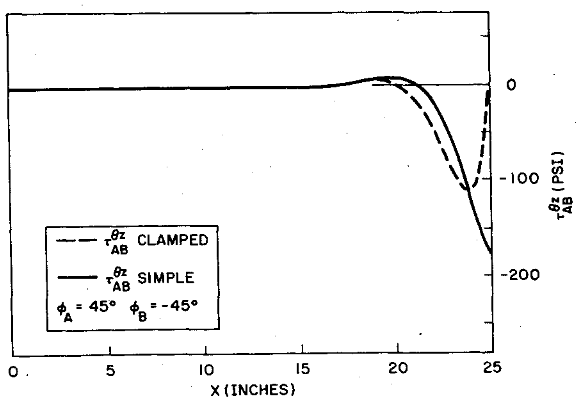

Fig. 3 Interlaminar circumferential shear stress distribution for both clamped and simply supported ends.

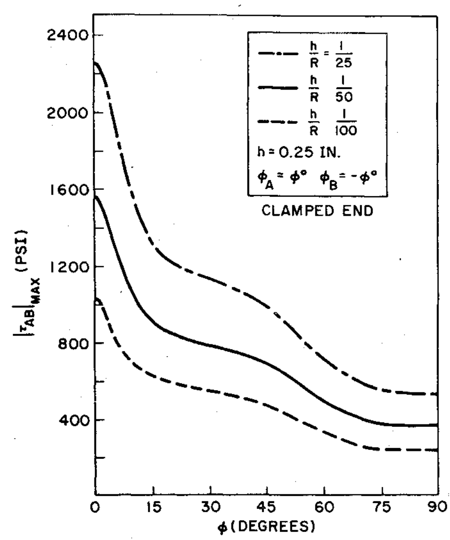

Fig. 5 Variation of $|\tau_{AB}|_{\max}$ with fiber orientation at a clamped end.

Similarly, from Fig. 4, it is seen that the interlaminar normal stresses fluctuate slightly near the end of the shell, and then remain constant at half the value (because $h_A = h_B$) between the pressures exerted on the inner and outer surface, namely (-50) psi.

It is well known that a bending boundary layer exists in an isotropic shell within a short distance from any load or structural discontinuity.[21] Farther away from any load or structural discontinuity the shell is in a state of membrane stress under most real-life loads. For a shell of revolution, the boundary-layer distance is usually taken as $4(R_\theta h)^{1/2}$. Very recently,[22,25] it has been shown that, for a specially orthotropic shell the bending boundary layer in the axial direction of a cylindrical shell is $4[(D_{11}/D_{22})^{1/2} Rh)]^{1/2}$ where

$$D_{ij} = \sum_{k=1}^{N} \frac{\bar{Q}_{ij}}{3}[h_k^3 - h_{k-1}^3]$$

where the terms are defined in Refs. 22, 23, and many others. In the shell being discussed (i.e., 2 laminae, $+45°/-45°$) $D_{11} = D_{22}$, hence the bending boundary layer is simply $4(Rh)^{1/2}$ and for the geometry of the shell it is 7.07 in, or ends at $x = 17.93$ in. Looking at Figs. 2-4, it is seen that all interlaminar shear stresses are zero at $x < 17.93$ in. and that the normal stress has reached a constant value. Physically, this confirms that the interlaminar shear stresses are caused by structural (or load) discontinuities, in the region of nonzero bending moments and shear resultants, and that the interlaminar shear stresses are zero in areas of membrane behavior only.

In all cases considered, the magnitude of the interlaminar shear stress, defined as

$$|\tau_{AB}| = [(\tau_{AB}^{xz})^2 + (\tau_{AB}^{\theta z})^2]^{1/2}$$

is maximum at the end of the shell ($x = 25$ in). The variation of this $|\tau_{AB}|_{\max}$ with fiber orientation for this $+\phi°/-\phi°$ angle ply shell were investigated for both clamped and simply supported edges. Figure 5 shows the former boundary condition and shows the effect of varying the radius of the shell. It is seen that the maximum shear stresses decrease with increased radius; also, the magnitude of the maximum values of interlaminar shear stresses are minimized when $\phi = 90°$, (a hoop wrap only). In the case of $\phi = 0°$ or $90°$, these shear stresses are at the shell midplane of a single layer orthotropic shell of thickness $(h_A + h_B)$ cna can be calculated by any orthotropic shell theory that involves transverse shear deformation.

A three-lamina composite shell also was investigated in which the baseline thickness is $h_A = h_B = h_C = 0.833$ in. Analogous to the two-lamina case, the interlaminar shear stresses are negligible farther from the clamped or simply supported edge than $4[(D_{11}/D_{22})^{1/2} Rh]^{1/2}$, the bending boundary layer.

Various stacking sequences were investigated such as $+\phi°/-\phi°/90°$, $+\phi°/90°/-\phi°$, $90°/+\phi°/-\phi°$, where the sequence refers to the *A-B-C* layers, going from the outer layer to inner layer. These alternatives produced no significant reduction in the interlaminar shear stresses for any particular stacking sequence, but the interlaminar normal stress at a clamped end was highly dependent upon the stacking sequence.[24]

The effects of material properties were studied through investigating the baseline case using fiberglas-epoxy with the following properties rather than the boron-epoxy:

$$E_{11}=6\times10^6 \text{ psi},\quad E_{22}=1.5\times10^6 \text{ psi},\quad \nu_{12}=0.25,$$

$$G_{12}=G_{13}.0.8\times10^6 \text{ psi},\quad G_{23}=0.6\times10^6 \text{ psi}$$

The case of a three-lamina shell of 90°/0°/90° construction was studied. The primary result is that the interlaminar shear stresses are greater for the glass epoxy material for both clamped and simply-supported end boundary conditions, than for the same shell using boron-epoxy.

VII. Conclusions

Methods of analysis have been developed for determining the interlaminar shear and normal stresses for laminated circular cylindrical shells of composite materials subjected to axially symmetric loads. Complete solutions have been obtained for both the two-lamina and three-lamina cases, and a computer program is available for ease of calculation.[24] These methods provide a) accurate solutions for shells of one, two- or three-laminae; b) considerable insight into the interlaminar stress fields of shells with any number of plies; and c) a checkpoint or baseline for finite element or finite difference solutions or other approximate solutions, of the same problems.

It is seen that the interlaminar shear stresses are confined to a region known as the bending boundary layer. In this region bending stress couples and transverse shear resultants also exist. Outside of the bending boundary layer a state of membrane stress and deformation exists, and interlaminar shear stresses are zero. Hence, inspection of interlaminar joints can be concentrated in the bending boundary-layer regions near to any load or structural discontinuity (such as an edge, change of section, etc.). The dimensions of the bending boundary layer are sized approximately herein. Knowledge of these characteristics can reduce inspection and acceptance problems and increase reliability of structural components.

Secondly, it is seen that for uniform loads, the magnitude of the interlaminar shear stress is a maximum at the clamped or simply supported edges. If the strength of the interlaminar matrix or bond material is exceeded, delamination can initiate at or near the edges.

The interlaminar stresses calculated herein, through the use of an accurate shell theory, include those arising from the analysis of a heterogeneous solid, as done by Pagano and Pipes,[11] and others.[13, 17-19] Thus, if one considered a cylindrical shell with a free end, subjected to an internal or external pressure, both shear and normal interlaminar stresses with exist at the free edge because of the relative in-plane and bending stiffnesses existing in the two or three laminae.

Similar studies need to be performed for laminated cylinders under in-plane loads, torsional loads, and combinations of all three load conditions, as well as shells of other configurations. Each parametric study will yield information that will enable the designer and analyst to understand and minimize the effects of interlaminar stresses. Such studies can employ the assumptions and approaches used herein.

Appendix

Several calculations were made to compare the result of previous methods of analysis with those presented herein. Results obtained from using methods from Ref. 21, Chap. 6, were compared to those presented herein for the case of an isotropic circular cylindrical shell is steel, for the baseline geometry and load (see Table 1).

The small difference arising between the methods are because the methods of Ref. 21 use classical assumptions, i.e., neglect transverse shear deformation which are perfectly applicable for thin shells of isotropic materials, which for this case of $h/R=1/50$, amounts to no more error than 1.6% (in M_x at the clamped end).

Table 1 Comparison of present methods for Ref. 21

Location	Ref. 21	2 lamina	3 lamina
Clamped end			
M_x (in.-lb/in.)	-93.3	-91.80	-91.78
Q_x (lb/in.)	-136.6	-134.9	-134.8
Simply supported end			
M_x	0	0	0
Q_x	68.3	68.00	67.97
Shell midlength			
w (in.)	0.00192	0.001923	0.001923

Table 2 Comparison of present methods with Ref. 3

	Kingsbury[3]	Two lamina	Three lamina
$\phi=0°w$ (in.)	0.340	0.03396	0.3397
$\phi=90°w$ (in.)	0.00192	0.001923	0.001923

Next, midspan deflections for a specially orthotropic shell of boron-epoxy were compared using the baseline geometry and load, between the present methods and those of Kingsbury[3] (see Table 2).

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